

PROJECTIVE MODULES AND INVOLUTIONS

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ABSTRACT. Let G be a finite group, and let $\Omega := \{t \in G \mid t^2 = 1\}$. Then Ω is a G -set under conjugation. Let k be an algebraically closed field of characteristic 2. It is shown that each projective indecomposable summand of the G -permutation module $k\Omega$ is irreducible and self-dual, whence it belongs to a real 2-block of defect zero. This, together with the fact that each irreducible kG -module that belongs to a real 2-block of defect zero occurs with multiplicity 1 as a direct summand of $k\Omega$, establishes a bijection between the projective components of $k\Omega$ and the real 2-blocks of G of defect zero.

Let G be a finite group, with identity element e , and let $\Omega := \{t \in G \mid t^2 = e\}$. Then Ω is a G -set under conjugation. In this note we describe the projective components of the permutation module $k\Omega$, where k is an algebraically closed field of characteristic 2. By a projective component we mean an indecomposable direct summand of $k\Omega$ that is also a direct summand of a free kG -module. We show that all such components are irreducible, self-dual and occur with multiplicity 1.

This gives an alternative proof of Remark (2) on p. 254 of [5], and strengthens Corollaries 3 through 7 of that paper. In addition, we can give the following quick proof of Proposition 8 in [5]:

Corollary 1. *Suppose that H is a strongly embedded subgroup of G . Then $k_H \uparrow^G \cong k_G \oplus [\oplus_{i=1}^s P_i]$ where $s \geq 0$ and the P_i are pairwise nonisomorphic self-dual projective irreducible kG -modules.*

Proof. That H is strongly embedded means that $|H|$ is even and $|H \cap H^g|$ is odd, for each $g \in G \setminus H$. Let $t \in H$ be an involution. Then clearly $C_G(t) \leq H$. So $k_H \uparrow^G$ is isomorphic to a submodule of $(k_{C_G(t)}) \uparrow^G$. Mackey's theorem implies that every component of $k_H \uparrow^G$, other than k_G , is a projective kG -module. Being projective, these modules must be components of $(k_{C_G(t)}) \uparrow^G$. The result now follows from Theorem 8. \square

Consider the wreath product $G \wr \Sigma$ of G with a cyclic group Σ of order 2. Here Σ is generated by an involution σ and $G \wr \Sigma$ is isomorphic to the semidirect product of the base group $G \times G$ by Σ . The conjugation action of σ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. The elements of $G \wr \Sigma$ will be written (g_1, g_2) , $(g_1, g_2)\sigma$ or σ .

We shall exploit the fact that kG is a $kG \wr \Sigma$ -module. For, as is well-known, kG is an $k(G \times G)$ -module via: $x \cdot (g_1, g_2) := g_1^{-1}xg_2$, for each $x \in kG$, and $g_1, g_2 \in G$. The action of Σ on kG is induced by the permutation action of σ on the

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distinguished basis G of kG : $g^\sigma := g^{-1}$, for each $g \in G$. Clearly σ acts as an involutory k -algebra anti-automorphism of kG . It follows that the actions of $G \times G$ and Σ on kG are compatible with the group relations in $G \wr \Sigma$.

By a *block* of kG , or a 2-block of G , we mean an indecomposable k -algebra direct summand of kG . Each block has associated to it a primitive idempotent in $Z(kG)$, a Brauer equivalence class of characters of irreducible kG -modules and a Brauer equivalence class, modulo 2, of ordinary irreducible characters of G . A block has defect zero if it is a simple k -algebra, and is real if it contains the complex conjugates of its ordinary irreducible characters. Theorem 8 establishes a bijection between the real 2-blocks of G that have defect zero and the projective components of $k\Omega$.

We could equally well work over a complete discrete valuation ring R of characteristic 0, whose field of fractions F is algebraically closed, and whose residue field $R/J(R)$ is k . So we use \mathcal{O} to indicate either of the commutative rings k or R .

All our modules are right-modules. We denote the trivial $\mathcal{O}G$ -module by \mathcal{O}_G . If M is an $\mathcal{O}G$ -module, we use $M \downarrow_H$ to denote the restriction of M to H . If H is a subgroup of G and N is an $\mathcal{O}H$ -module, we use $N \uparrow^G$ to denote the induction of N to G . Whenever $g \in G$, we write \underline{g} for $(g, g) \in G \times G$, and we set $\underline{X} := \{\underline{x} \mid x \in X\}$, for each $X \subset G$. Other notation and concepts can be found in a standard textbook on modular representation theory, such as [1] or [4].

If B is a block of $\mathcal{O}G$, then so too is $B^\sigma = \{x^\sigma \mid x \in B\}$. We call B a real block if $B = B^\sigma$. Our first result describes the components of $\mathcal{O}G$ as $\mathcal{O}G \wr \Sigma$ -module.

Lemma 2. *There is an indecomposable decomposition of $\mathcal{O}G$ as $\mathcal{O}G \wr \Sigma$ -module:*

$$\mathcal{O}G = B_1 \oplus \dots \oplus B_r \oplus (B_{r+1} + B_{r+1}^\sigma) \oplus \dots \oplus (B_{r+s} + B_{r+s}^\sigma).$$

Here B_1, \dots, B_r are the real 2-blocks and $B_{r+1}, B_{r+1}^\sigma, \dots, B_{r+s}, B_{r+s}^\sigma$ are the non-real 2-blocks of G .

Proof. This follows from the well-known indecomposable decomposition of $\mathcal{O}G$, as an $\mathcal{O}(G \times G)$ -module, into a direct sum of its blocks, and the fact that $B_i^\sigma = B_i$ for $i = 1, \dots, r$, and $B_{r+j}^\sigma = B_{r+j}^\sigma$ for $j = 1, \dots, s$. \square

An obvious but useful fact is that $\mathcal{O}G$ is a permutation module:

Lemma 3. *The $\mathcal{O}G \wr \Sigma$ -module $\mathcal{O}G$ is isomorphic to the permutation module $(\mathcal{O}_{\underline{G} \times \Sigma}) \uparrow^{G \wr \Sigma}$.*

Proof. The elements of G form a $G \wr \Sigma$ -invariant basis of $\mathcal{O}G$. Moreover if $g_1, g_2 \in G$, then $g_2 = g_1 \cdot (g_1, g_2)$. So G is a transitive $G \wr \Sigma$ -set. The stabilizer of $e \in \mathcal{O}G$ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$. The lemma follows from these facts. \square

Let C be a conjugacy class of G . Set $C^\sigma := \{c \in G \mid c^{-1} \in C\}$. Then C^σ is also a conjugacy class of G , and $C \cup C^\sigma$ can be regarded as an orbit of $\underline{G} \times \Sigma$ on the $G \wr \Sigma$ -set G . As such, the corresponding permutation module $\mathcal{O}(C \cup C^\sigma)$ is a $\mathcal{O}\underline{G} \times \Sigma$ -direct summand of $\mathcal{O}G$. If $C = C^\sigma$, we call C a real class of G . In this case for each $c \in C$ there exists $x \in G$ such that $c^x = c^{-1}$. The point stabilizer of c in $\underline{G} \times \Sigma$ is $\underline{C_G(c)} < \underline{x}\sigma >$. So $\mathcal{O}C \cong (\mathcal{O}_{\underline{C_G(c)} < \underline{x}\sigma >}) \uparrow^{\underline{G} \times \Sigma}$. If $C \neq C^\sigma$, we call C a nonreal class of G . In this case the point stabilizer of $c \in C \cup C^\sigma$ in $\underline{G} \times \Sigma$ is $\underline{C_G(c)}$. So $\mathcal{O}(C \cup C^\sigma) \cong (\mathcal{O}_{\underline{C_G(c)}}) \uparrow^{\underline{G} \times \Sigma}$.

Suppose now that $\overline{C_1}, \dots, \overline{C_t}$ are the real classes of G and that $C_{t+1}, C_{t+1}^\sigma, \dots, C_{t+u}, C_{t+u}^\sigma$ are the nonreal classes. Then we have:

Lemma 4. *There is a decomposition of $\mathcal{O}G$ as an $\mathcal{O}\underline{G} \times \Sigma$ -permutation module:*

$$\mathcal{O}G = \mathcal{O}C_1 \oplus \dots \oplus \mathcal{O}C_t \oplus \mathcal{O}(C_{t+1} \cup C_{t+1}^o) \oplus \dots \oplus \mathcal{O}(C_{t+u} \cup C_{t+u+1}^o).$$

Proof. This follows from Lemma 3 and the discussion above. \square

By a quasi-permutation module we mean a direct summand of a permutation module. Our next result is Lemma 9.7 of [1]. We include a proof for the convenience of the reader.

Lemma 5. *Let M be an indecomposable quasi-permutation $\mathcal{O}G$ -module and suppose that H is a subgroup of G such that $M \downarrow_H$ is indecomposable. Then there is a vertex V of M such that $V \cap H$ is a vertex of $M \downarrow_H$. If H is a normal subgroup of G , then this is true for all vertices of M .*

Proof. Let U be a vertex of M . As $\mathcal{O}_U \mid M \downarrow_U$ we have $\mathcal{O}_{U \cap H} \mid (M \downarrow_H) \downarrow_{U \cap H}$. But $U \cap H$ is a vertex of $\mathcal{O}_{U \cap H}$. So Mackey's Theorem implies that there exists a vertex W of $M \downarrow_H$ such that $U \cap H \leq W$.

As $M \downarrow_H$ is a component of the restriction of M to H , Mackey's Theorem shows that there exists $g \in G$ such that $W \leq U^g \cap H$. Now U^g is a vertex of M . So by the previous paragraph, and the uniqueness of vertices of $M \downarrow_H$ up to H -conjugacy, there exists $h \in H$ such that $U^g \cap H \leq W^h$. Comparing cardinalities, we see that $W = U^g \cap H$. So $U^g \cap H$ is a vertex of $M \downarrow_H$.

Suppose that H is a normal subgroup of G . Then $U \cap H \leq W$ and $W = U^g \cap H = (U \cap H)^g$ imply that $U \cap H = W$. \square

R. Brauer showed how to associate to each block of $\mathcal{O}G$ a G -conjugacy class of 2-subgroups, its so-called defect groups. It is known that a block has defect zero if and only if its defect groups are all trivial. J. A. Green showed how to associate to each indecomposable $\mathcal{O}G$ -module a G -conjugacy class of 2-subgroups, its so-called vertices. He also showed how to identify the defect groups of a block using its vertices as an indecomposable $\mathcal{O}(G \times G)$ -module.

Corollary 6. *Let B be a block of $\mathcal{O}G$ and let D be a defect group of B . If B is not real then \underline{D} is a vertex of $B + B^o$, as $\mathcal{O}G \wr \Sigma$ -module. If B is real, then there exists $x \in N_G(D)$, with $x^2 \in D$, such that $\underline{D} < \underline{x}\sigma >$ is a vertex of B , as $\mathcal{O}G \wr \Sigma$ -module. In particular, Σ is a vertex of $B + B^o$ if and only if B is a real 2-block of G that has defect zero.*

Proof. J. A. Green showed in [2] that \underline{D} is a vertex of B , when B is regarded as an indecomposable $\mathcal{O}(G \times G)$ -module. Suppose first that B is not real. Then $B + B^o = (B \downarrow_{G \times G}) \uparrow^{G \wr \Sigma}$, for instance by Corollary 8.3 of [1]. It follows that $B + B^o$ has vertex \underline{D} , as an indecomposable $\mathcal{O}G \wr \Sigma$ -module.

Suppose then that $B = B + B^o$ is real. Lemma 3 shows that B is $\underline{G} \times \Sigma$ -projective. So we may choose a vertex V of B such that $V \leq \underline{G} \times \Sigma$. Moreover, B is a quasi-permutation $\mathcal{O}G \wr \Sigma$ -module, and its restriction to the normal subgroup $G \times G$ is indecomposable. Lemma 5 then implies that $V \cap (G \times G) = V \cap \underline{G}$ is a vertex of $B \downarrow_{G \times G}$. So by Green's result, we may choose D so that $V \cap \underline{G} = \underline{D}$. Now $G \times G$ has index 2 in $G \wr \Sigma$. So Green's indecomposability theorem, and the fact that $B \downarrow_{G \times G}$ is indecomposable, implies that $V \not\leq (G \times G)$. It follows that there exists $x \in N_G(D)$, with $x^2 \in D$, such that $V = \underline{D} < \underline{x}\sigma >$.

If B has defect zero, then $D = \langle e \rangle$. So $x^2 = e$. In this case, $\langle \underline{x}\sigma \rangle = \Sigma^{(e,x)}$ is $G \wr \Sigma$ -conjugate to Σ . So Σ is a vertex of B . Conversely, suppose that Σ is a vertex

of $B + B^\circ$. The first paragraph shows that B is a real block of G . Moreover B has defect zero, as $\Sigma \cap \underline{G} = \langle e \rangle$. \square

We quote the following result of Burry, Carlson and Puig [4, 4.4.6] on the Green correspondence:

Lemma 7. *Let $V \leq H \leq G$ be such that V is a p -group and $N_G(V) \leq H$. Let f denote the Green correspondence with respect to (G, V, H) . Suppose that M is an indecomposable $\mathcal{O}G$ -module such that $M \downarrow_H$ has a component N with vertex V . Then V is a vertex of M and $N = f(M)$.*

We can now prove our main result. Part (ii) is Remark (2) on p. 254 of [5], but our proof is independent of the proof given there.

Theorem 8. (i) *Let $t \in G$, with $t^2 = e$. Suppose that P is an indecomposable projective direct summand of $(\mathcal{O}_{C_G(t)})^\uparrow^G$. Then P is irreducible and self-dual and occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_G(t)})^\uparrow^G$. In particular P belongs to a real 2-block of G that has defect zero.*

(ii) *Suppose that M is a projective indecomposable $\mathcal{O}G$ -module that belongs to a real 2-block of G that has defect zero. Then there exists $s \in G$, with $s^2 = e$, such that M is a component of $(\mathcal{O}_{C_G(s)})^\uparrow^G$. Moreover, s is uniquely determined up to conjugacy in G .*

Proof. If $t = e$ then $P = \mathcal{O}_G$. So P is irreducible and self-dual. The assumption that P is projective and the fact that $\dim_{\mathcal{O}}(P) = 1$ implies that $|G|$ is odd. So all blocks of $\mathcal{O}G$, in particular the one containing P , have defect zero.

Now suppose that $t \neq e$. Let T be the conjugacy class of G that contains t . The permutation module $\mathcal{O}T$ is a direct summand of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$. Regard P as an $\mathcal{O}\underline{G}$ -module. Let $I(P)$ be the inflation of this module to $\underline{G} \times \Sigma$. Then $I(P)$ is a component of $\mathcal{O}T$. As Σ is contained in the kernel of $I(P)$, and P is a projective $\mathcal{O}G$ -module, it follows that $I(P)$ has vertex Σ as an indecomposable $\mathcal{O}\underline{G} \times \Sigma$ -module.

By Lemma 2, and the Krull-Schmidt theorem, there exists a 2-block B of G such that $I(P)$ is a component of the restriction $(B + B^\circ) \downarrow_{\underline{G} \times \Sigma}$. An easy computation shows that $N_{G \wr \Sigma}(\Sigma) = \underline{G} \times \Sigma$. It then follows from Lemma 7 that $(B + B^\circ)$ has vertex Σ and also that $I(P)$ is the Green correspondent of $(B + B^\circ)$ with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$. We conclude from Corollary 6 that B is a real 2-block of G that has defect zero.

Let \hat{B} be the 2-block of $G \wr \Sigma$ that contains B . Then \hat{B} is real and has defect group Σ . Let \hat{A} be the Brauer correspondent of \hat{B} . Then \hat{A} is a real 2-block of $\underline{G} \times \Sigma$ that has defect group Σ . Now $\hat{A} = A \otimes \mathcal{O}\Sigma$, where A is a real 2-block of $\mathcal{O}\underline{G}$ that has defect zero. In particular A has a unique indecomposable module, and this module is projective, irreducible and self-dual. Corollary 14.4 of [1] implies that $I(P)$ belongs to \hat{A} . So P belongs to A . We conclude that P is irreducible and self-dual and belongs to a real 2-block of G that has defect zero.

Now B occurs with multiplicity 1 as a component of $\mathcal{O}G$, and $I(P)$ is the Green correspondent of B with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$. So $I(P)$ has multiplicity 1 as a component of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$. It follows that P occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_G(t)})^\uparrow^G$, and with multiplicity 0 as a component of $(\mathcal{O}_{C_G(r)})^\uparrow^G$, for $r \in G$ with $r^2 = e$, but r not G -conjugate to t . This completes the proof of part (i).

Let R be a real 2-block of G that has defect zero. Then R has vertex Σ as indecomposable $\mathcal{O}G \wr \Sigma$ -module. So its Green correspondent $f(R)$, with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$, is a component of the restriction of $\mathcal{O}G$ to $\underline{G} \times \Sigma$ that has vertex Σ . Lemma 4 and the Krull-Schmidt theorem imply that $f(R)$ is isomorphic to a component of $\mathcal{O}(C \cup C^o)$, for some conjugacy class C of G . Now Σ is a central subgroup of $\underline{G} \times \Sigma$. So Σ must be a subgroup of the point stabilizer of $C \cup C^o$ in $\underline{G} \times \Sigma$. It follows that $s^2 = e$, for each $s \in C$. Let N denote the restriction of $f(R)$ to \underline{G} , and consider N as an $\mathcal{O}G$ -module. We have just shown that N is a component of $(\mathcal{O}_{C_G(s)})^\uparrow^G$. Argueing as before, we see that N is an indecomposable projective $\mathcal{O}G$ -module that belongs to a real 2-block of G that has defect zero.

The last paragraph establishes an injective map between the real 2-blocks of G that have defect zero and certain projective components of $\mathcal{O}\Omega$. As each block of defect zero contains a single irreducible $\mathcal{O}G$ -module, this map must be onto. It follows that the module M in the statement of the theorem is a component of some permutation module $(\mathcal{O}_{C_G(s)})^\uparrow^G$, where $s \in G$ and $s^2 = e$. The fact that s is determined up to G -conjugacy now follows from the last statement of the proof of part (i). This completes the proof of part (ii). \square

It is possible to simplify the above proof by showing that if B is a real 2-block of G that has defect zero, then its Green correspondent, with respect to $(G \wr \Sigma, \Sigma, \underline{G} \times \Sigma)$ is M^{Fr} , where M^{Fr} is the Frobenius conjugate of the unique irreducible $\mathcal{O}G$ -module that belongs to B .

Corollary 9. *Let $\Omega = \{t \in G \mid t^2 = e\}$. Then there is a bijection between the real 2-blocks of G that have defect zero and the projective components of $\mathcal{O}\Omega$.*

Here is a sample application. It was suggested to me by G. R. Robinson.

Corollary 10. *Let $n \geq 1$ and let t be an involution in the symmetric group Σ_n . If $n = m(m+1)/2$ is a triangular number, and t is a product of $\lfloor \frac{m^2+1}{4} \rfloor$ commuting transpositions, then there is a single projective irreducible $\mathcal{O}\Sigma_n$ -module, and this module is the unique projective component of $(\mathcal{O}_{C_{\Sigma_n}(t)})^\uparrow^{\Sigma_n}$. For all other values of n or nonconjugate involutions t , the modules $(\mathcal{O}_{C_{\Sigma_n}(t)})^\uparrow^{\Sigma_n}$ are projective free.*

Proof. We give a proof of the following result in [3, Corollary 8.4]: Let G be a finite group, let B be a real 2-block of G of defect zero, and let χ be the unique irreducible character in B . Then there exists a 2-regular conjugacy class C of G such that $C = C^o$, $|C_G(c)|$ is odd, for $c \in C$, and $\chi(c)$ is nonzero, modulo a prime ideal containing 2. Moreover, there exists an involution $t \in G$ such that $c^t = c^{-1}$, and for this t we have $\langle \chi_{C_G(t)}, 1_{C_G(t)} \rangle = 1$. The existence of t was shown in [5]. The identification of t using the class C was first shown by R. Gow (in unpublished work).

Suppose that $(\mathcal{O}_{C_{\Sigma_n}(t)})^\uparrow^{\Sigma_n}$ has a projective component. Then Σ_n has a 2-block of defect zero, by Theorem 8. The 2-blocks of Σ_n are indexed by triangular partitions $\mu = [m, m-1, \dots, 2, 1]$, where m ranges over those natural numbers for which $n - m(m+1)/2$ is even. Moreover, the 2-block corresponding to μ has defect zero if and only if $n = m(m+1)/2$. In particular, we can assume that $n = m(m+1)/2$, for some $m \geq 1$.

Let B be the unique 2-block of Σ_n that has defect zero, let χ be the unique irreducible character in B and let $g \in \Sigma_n$ have cycle type $\lambda = [2m-1, 2m-5, \dots]$. Then $|C_{\Sigma_n}(g)|$ is odd. As the parts of λ are the ‘‘diagonal hooklengths’’ of μ ,

the Murnaghan-Nakayama formula shows that $\chi(g) = 1$. Now λ has $\lfloor (m-1)/2 \rfloor$ nonzero parts. So g is inverted by an involution t that is a product of $(n - \lfloor (m-1)/2 \rfloor)/2 = \lfloor \frac{m^2+1}{4} \rfloor$ commuting transpositions. It follows from Theorem 8 and the previous paragraph that the unique irreducible projective B -module occurs with multiplicity 1 as a component of $(\mathcal{O}_{C_{\Sigma_n}(t)})^{\uparrow \Sigma_n}$. The last statement of the Corollary now follows from Theorem 8. \square

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